Secondary school students' beliefs and attitudes about errors in mathematics and the formation of their MWS

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Abstract. *The present article describes the first part of a survey which investigates lower secondary school students' beliefs about formative assessment in mathematics. Based on students' beliefs related to formative assessment, this research is a first attempt to propose an appropriate MWS for the secondary education students, regarding to the handling of the error in mathematics classroom. Four hundred twenty eight lower secondary school students completed a questionnaire about the formative assessment. The present article concentrates on students' beliefs about the use of mathematical errors and how the use of errors in mathematics can create a suitable MWS. The findings of the study reveal that the students focus on the suitable MWS rather than their personal MWS for the process of elaborating their errors. They consider the role of the teacher in the formative use of the mathematical error as the most significant element. Furthermore, the interaction between the students about their errors is essential for them.*

Keywords: formative assessment, mathematical error, attitudes, beliefs, conceptions, MKS (mathematical working space)

Sunto. *Il presente articolo descrive la prima parte di uno studio che indaga sulle credenze degli studenti della scuola secondaria circa la valutazione formativa in matematica. Sulla base delle credenze degli studenti relative alla valutazione formativa, questa ricerca è un primo tentativo di proporre un adeguato MWS (spazio di lavoro matematico) per gli studenti dell'istruzione secondaria, per quanto riguarda la gestione dell'errore di matematica in aula. Quattrocentoventotto studenti della scuola secondaria hanno completato un questionario sulla valutazione formativa. Il presente articolo si concentra sulle credenze degli studenti circa l'uso di errori matematici e come l'uso di errori in matematica sia in grado di creare un adeguato MWS. I risultati dello studio rivelano che gli studenti si concentrano solo sugli MWS adatti, piuttosto che sui loro MWS personali per il processo di elaborazione dei loro errori. Essi considerano il ruolo del docente nell'uso formativo dell'errore matematico come l'elemento più significativo. Inoltre, l'interazione tra gli studenti sui loro errori è secondo loro essenziale.*

Parole chiave: valutazione formativa, errore matematico, atteggiamenti, credenze, MWS (spazio di lavoro matematico)

Resumen. *El presente artículo describe la primera parte de un estudio que indaga sobre las creencias de los estudiantes de secundaria en relación con la evaluación* *formativa en matemática. Sobre la base de las creencias de los estudiantes en relación con la evaluación formativa, esta investigación es un primer intento de proponer un adecuado MWS (espacio de trabajo matemático) para los estudiantes de instrucción secundaria, por lo que respecta la gestión del error de matemática en el aula. Cuatrocientos veintiocho estudiantes de la escuela secundaria completaron un cuestionario sobre la evaluación formativa. El presente artículo se centra en las creencias de los estudiantes en referencia al uso de errores matemáticos y de cómo el uso del error en matemática ayude a crear un adecuado MWS. Los resultados del estudio revelan que los estudiantes se concentran únicamente en los MWS adecuados, y no en los MWS personales por el proceso de elaboración de sus errores. Ellos consideran el papel del docente en el uso formativo del error matemático como elemento de mayor significación. Además, la interacción entre los estudiantes y sus errores es según ellos esencial.*

Palabras clave: evaluación formativa, error matemático, actitud, creencias, MWS (espacio de trabajo matemático)

1. Introduction

Mathematics is a multi-functional and multi-disciplinary subject in school. This is the reason why the teachers need to be aware of their crucial position in school and their need to reflect on difficulties and mistakes, in order to find the causes of them and to plan the interventions for remedial programming, through efficacy strategies and tools of formative assessment. Recent international research (e.g. Eurydice, 2012; OECD, 2012) have determined five main difficulties in mathematics learning. One of them highlights the incorrect use of formative assessment and the need to introduce strategies of teaching and learning individualization (OECD, 2005; Weeden, Winter, & Broadfoot, 2002).

In fact, formative assessment (with its diagnostic function) allows calibrating the instructional strategies and differentiating them according to learners' needs. Formative assessment is essential in order to take valid and meaningful decisions to design activities, tools, materials and technologies, differentiated by learning levels and styles (Weeden, Winter, & Broadfoot, 2002). Moreover, formative assessment allows identifying the "critical steps" of mathematics and it also offers a clearer view of students' learning problems and addresses towards the most appropriate strategies to support and motivate students' learning. For this reason, Heritage (2013) highlights the need to investigate the mathematicians' beliefs and misconceptions about assessment in the classroom. He also, claims that it is essential to analyse learning activities in the classroom, investigating on teachers' rationales behind learning difficulties in mathematics in order to plan adequate interventions for remedial programming (Heritage, 2013).

The last years a lot of efforts have been done in research of mathematics education on models and modelling of mathematical works. All these researches will concentrate in this paper on the model of MWS, which was the object of multiple researches concerning the cognitive and epistemological analysis of different mathematical concepts. However, it seems that the model of Mathematical Working Space (MWS) does not take explicit account of the research in the affective domain in Mathematics education. Nevertheless, students' (and teachers') beliefs, attitudes and conceptions about the nature of mathematics, their skills in mathematics, the errors in mathematics, the causes of errors in mathematics and also the role of representations in teaching and learning of mathematics, can clearly contribute to the formation of three vertical axis (i.e. semiotic genesis, instrumental genesis, discursive genesis). Thus, the present study aims to enrich the model of MWS, emphasizing in the affective domain in Mathematics education.

2. Theoretical framework

2.1. Formative assessment in mathematics

Definitions and purpose of formative assessment

> Formative assessment, including *diagnostic testing*, is a range of formal and informal assessment procedures employed by teachers during the learning process in order to modify teaching and learning activities to improve student attainment (Crooks, 2001). It typically involves qualitative feedback (rather than scores) for both student and teacher that focuses on the details of content and performance (Huhta, 2010). It is commonly contrasted with *summative assessment*, which seeks to monitor educational outcomes, often for purposes of external accountability (Shepard, 2005). ("Formative assessment", n.d.)

Among the many different definitions of *formative assessment* (FA) provided by different researchers, common points occur to be emphasized in them. Many of these definitions put the teacher – student relation in the centre of the assessment process. For instance, Black and William (1998) define FA as the method that represents all classroom activities that are performed in classroom settings by either educators and/or their students and employ both of them in the process. This definition of FA highlights the interaction between the teacher and the student during the teaching and learning process. Other definitions about FA highlight its role for modifying the teaching and learning process based on the students' performance. For example, Van De Walle, Karp, and Bay-Williams (2013) claim that FA is "an along the way evaluation that monitors who is learning and who is not and helps teachers to form the next lesson" (p. 5).

A definition that combines all the previously stressed points is the one provided by Popham (2008, p. 5) and it is accepted by the Formative Assessment for Teachers and Students (FAST) group as the most accessible to educators (Clark, 2011). According to this definition, FA is "a process used by teachers and students during instruction that provides feedback to adjust ongoing teaching and learning to improve students' achievement of intended instructional outcomes" (Popham, 2008, p. 5).

The formative use of mathematical errors

The use of students' errors is an important dimension of FA, as it helps the teachers modify their techniques for helping the students correcting them, but also the students in identifying their weaknesses and try overcoming them. Wragg (2001) supports that "if students learn from their assessment, then the correction of errors and the discussion of what they have done is essential" (p. 74).

In fact, the identification of mistakes helps teachers decide how to identify and meet pupils' learning needs and how to use their teaching time and their resources (Kyriakides, 1999). The reason on which the teachers attribute the errors will affect their decisions for their future intervention teaching techniques. In fact, the research community argues that the mostly important errors are due to epistemological obstacles (Brousseau, 1998) or to didactical obstacles (Brousseau, 1990). Therefore, the students' errors can have a formative use, as the teachers can exploit this information for modifying their future actions (Gagatsis & Kyriakides, 2000). Thus, decisions about the next learning steps follow from the formative identification of pupils' errors (Desforges, 1989). And this is particularly important, because a teaching plan which is organized in such a way might help teachers to plan class and individual programs of work according to the different performance levels of the pupils (Gagatsis & Kyriakides, 2000).

This article focuses on the students' beliefs concerning to the mathematical error. In specific, it investigates how the students believe that the teacher has to use the mathematical error in the class. Based on these beliefs, we tried to suggest an "appropriate" MWS for the secondary school students, as regards the handling of the mathematical error in this level of education.

In this paper, we focus on the dimension of the use of error in mathematics, because the error is the "heart" of learning mathematics. Mathematics' teaching and understanding comes from the errors in mathematics, because the errors cause discussion, communication and feedback between the subjects in mathematics' classroom. In a class of mathematics without errors, the students couldn't develop their mathematical thinking, because the absence of the interaction that the mathematical error causes between teacher-students and student-student deprives students to develop the semiotic and discursive genesis.

Although formative assessment can be conducted through different techniques (e.g. self-assessment, peer-assessment, feedback or use of error) we choose to deal with the use of mathematical error, because it includes feedback technique, self-assessment and peer-assessment technique. However, this study takes into account the students' beliefs about the use of error based on three dimensions: (a) use of error by the teacher (feedback technique), (b) use of error by the student who is wrong (self-assessment technique), and (c) use of error between the students (feedback and peer-assessment technique). Focused on the above three dimensions, we try to find potential consistencies in the students' beliefs, in order to propose an "appropriate" MWS, as the students perceive it, that helps them to increase their understanding in mathematics. More specifically, we will try to prioritize the above dimensions of the handling of error in mathematics, according to their contribution in the students' understandings in mathematics. For these goals of the study, we take into account the students' answers in the 25 statements of the questionnaire. However, the research in the mathematical error is not limited in the aforementioned three dimensions, but there are research concerning the causes of the mathematical error (e.g. Gagatsis & Kyriakides, 2000).

2.2. The affective domain of research in mathematics education

The second part of the theoretical framework, concerns the affective domain of research in mathematics education. Key terms in this research are attitudes, beliefs and conceptions.

As it comes from the literature, there are various opinions concerning the notion of "beliefs". According to Goldin (1999), a belief may be "the multiply encoded cognitive configuration to which the holder attributes a high value, usually a truth value, including associated warrants" (Goldin, 1999, as cited in Presmeg, 2002, p. 293). Cooney (1999), asserts that a belief is "a cluster of dispositions to do various things under various circumstances" (Cooney, 1999, as cited in Presmeg, 2002, p. 293), which leads to the acceptance that "different circumstances may evoke different clusters of beliefs" (Presmeg, 2002, p. 293). It is widely accepted that beliefs are the individual's personal cognitions, theories and conceptions that one forms for subjective reasons. Their nature is partly logical and partly emotional. According to Mc Leod (1992) "beliefs are largely cognitive in nature, and are developed over a relatively long period of time" (p. 579).

Many researchers use attitudes as a term which includes beliefs about mathematics and about self. Mc Leod (1992) accepts that attitudes "refer to affective responses that involve positive or negative feelings of moderate intensity and reasonable stability" (p. 581); they may appear as a result of the automation "of a repeated emotional reaction to mathematics" (p. 581) or of "the assignment of an already existing attitude to a new but related task" (p. 581). However, to address the varying terminology about knowledge, beliefs, belief systems, and belief clusters more efficiently, Thompson (1992) invoked conceptions "as a more general mental structure, encompassing beliefs, meanings, concepts, propositions, rules, mental images, preferences, and the like" (p. 130).

A "conception" is a mental construction or representation of reality (Kelly, 1991), communicated in language or metaphors (Lakoff & Johnson, 2003) and which explains complex and difficult categories of experience (White, 1994) such as assessment. Furthermore, conceptions represent different categories of ideas held by teachers behind their descriptions of how educational things are experienced (Pratt, 1992). Thus, conceptions act as a framework though which a teacher views, interprets and interacts with the teaching environment (Marton, 1981).

Students' beliefs about mathematics and assessment

Over the last two decades the role of beliefs, as well as the role of knowledge, in cognitive processes has been recognised. In particular, students' general beliefs about the nature and acquisition of knowledge, namely epistemological beliefs, have been investigated regarding their influence on text comprehension and metacomprehension (Kardash & Howell, 2000), problem solving (Schraw, Dunkle, & Bendixen, 1995), and conceptual change (Mason, 2000). Students' beliefs have been investigated not only as general convictions, but also as convictions about knowing and learning in specific domains, including mathematics (De Corte, Op't Eynde, & Verschaffel, 2002). Schoenfeld (1983) pointed out the existence of a system of beliefs that drives students' behaviour when trying to solve mathematical problems, since problem solving performance cannot be seen as purely cognitive. He revealed that students' beliefs about what is useful in learning maths affects the cognitive resources available to them when learning in this domain, making a large portion of stored information inaccessible when the beliefs impede rather than facilitate understanding. Furthermore, students' conceptions of assessment are of particular importance because assessment has a significant impact on the quality of learning (Ramsden, 1997).

The research literature on students' conceptions of assessment is not vast, and is largely focused on tertiary or higher education students (Struyven, Dochy, & Janssens, 2005). Review of the empirical literature on students' conceptions of the purposes of assessment has identified four major purposes, some of which can be matched to teachers' conceptions of assessment. Students are reported as conceiving of assessment as (a) improving achievement, (b) a means for making them accountable, (c) being irrelevant, and (d) being enjoyable.

2.3. Mathematical working space (MWS)

The model of Mathematical Working Space (MWS) was first developed for the geometry (Houdement & Kuzniak, 2006; Kuzniak, 2006), while a first approach to the concept and the MWS structure was made by Kuzniak (2011), based on the Geometrical Working Space (GWS).

According to Kuzniak (2011) the figural genesis, in the model of Geometrical Working Space and the visualization should be modified and re-interpreted by the processes of semiotic representation associated with the mathematical topic studied at a time. For this reason, the figural genesis converted to semiotic genesis, because the semiotic representations are the "heart" of mathematics and the basic principle of cognitive processes concerning the mathematics understanding. Below, the Figure 1 shows the model of Mathematical Working Space (MWS), as proposed by Kuzniak (2011) and Kuzniak and Richard (2014).

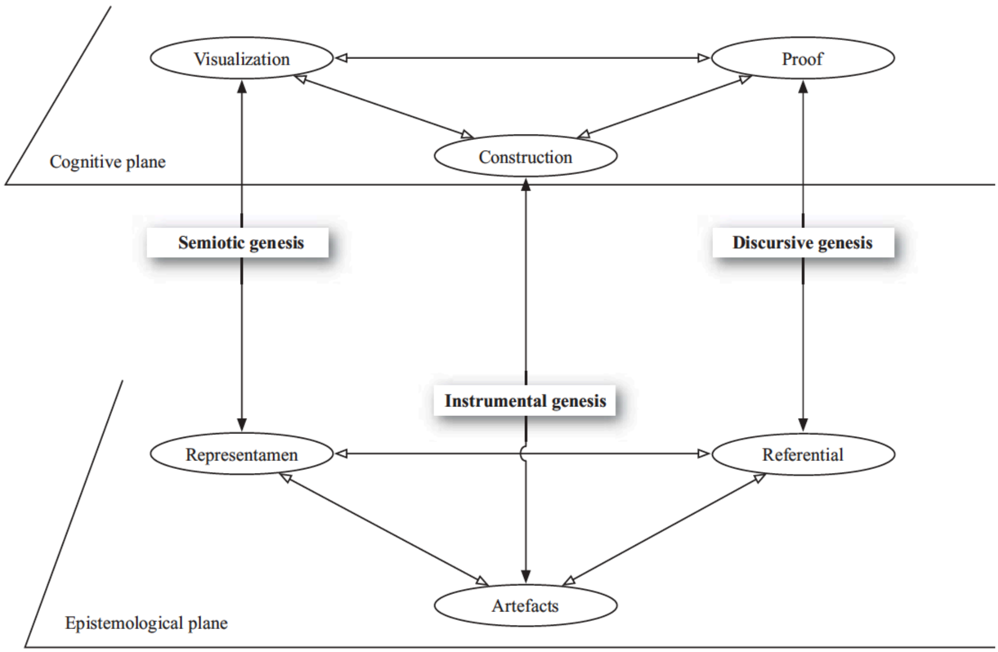

Figure 1. The model of Mathematical Working Space (MWS) (Kuzniak & Richard, 2014, p. 21).

The MWS consists of two components: epistemological and cognitive. In order for the epistemological component to be also applied to other mathematical areas except geometry, the model should be modified in MWS and be based on the concept of the sign or representation, which is the fundamental component of mathematical work, as introduced by Peirce (1839–1914). The semiotic genesis associated with semiotic representations of mathematical objects (provides the mathematical objects in tangible objects).

As regards the cognitive dimension of MWS, a visualization process linked to the figures' processing (mentally figures) and the intuition. In mathematics, the signs are usually visual in nature. Even the algebraic notations need to be visible either mentally or in written form.

According to Kuzniak, Tanguay, and Elia (2016) the structure of the epistemological and cognitive aspects within the MWS model aims to provide a tool for the study of mathematical work in which students and teachers are effectively engaged during mathematics notions and it allows the analysis of the mathematical activity of individuals dealing with mathematical problems. Based on the MWS model in the analysis of the mathematical activity we can observe the development of a concept, as a process of bridging the epistemological and the cognitive perspectives.

Three dimensions are discriminated in the model: semiotic, instrumental, and discursive genesis. Considerable research work has been done on the dimension of MWS referring to the semiotic representations, semiotic genesis, and visualization. This research approached several mathematical areas, such as geometry, arithmetic, probability and statistics, and other mathematical concepts at different levels of education (primary, secondary, university). These studies (Barrera, 2013; Gagatsis et al., 2016; Mora et al., 2016; Panero, Arzarello, & Sabena, 2016; Santos-Trigo, Moreno-Armella, & Camacho-Machín, 2016) promoted the important role of the semiotic aspects on the students' MWS. Gagatsis and his colleagues (2016) studied how the representational flexibility is developed in fractions and decimal numbers addition in connection with the MWS model. The findings of their research show that the axis of semiotic genesis in fractions and decimal numbers is not automatic, but requires a long process of small steps of development. Therefore, further research is needed to better understand how the visualization and semiotic representations can be used in a MWS, in order to achieve effective learning in mathematics.

The theory of the instrumental genesis developed by psychologists regardless of their socio-cultural approach in the mid 90's and many researchers (Mariotti, 2002) consider that it illuminates the process of internalizing the tools and the semiotic mediation they perform. It mainly refers to the artefacts, giving them specific capabilities and specific uses; however, the conversion of an artefact to a cognitive tool is succeeded via a complicated process, which does not necessarily lead to a deeper understanding of concepts (Guin & Trouche, 1998). Moreover, the instrumental genesis depends on the tasks assigned to each student. The development of the concepts by the students as a result of their interaction with the learning environment raises an important issue concerning the interpretation of the phenomena observed on a computer screen. As the different coordinated patterns are sequentially formed, so the relationship between the user and the artefact evolves: this process is called "instrumental genesis" (Mariotti, 2002).

The discursive genesis of MWS enables us to follow a theoretical approach to the analysis of the obstacles and difficulties encountered by students in their reasoning in a mathematical concept. This reasoning is produced in various formal or non-formal activities on the mathematical concept taught, as well as providing explanations about the meaning of the concept. Within this genesis we are able to identify and interpret the perceptions of students and their mistakes as they think about this concept. According to Kuzniak and Richard

(2014) the discursive genesis of the proof used by the properties combined together on the theoretical referential in order to put them in service to the mathematical reasoning and to a non-exclusively iconic, graphic, or instrumented validation.

3. Methodology

3.1 Participants

This study is a part of a survey in the context of a European Program,[1] named Formative Assessment of Mathematics Teaching and Learning (FAMT&L) (more details about the program can be found in Michael-Chrysanthou, Gagatsis, & Vannini, 2014). The particular study is also the first part of a doctoral thesis of Theodora Christodoulou in preparation, which aims to propose a model describing lower secondary school students' beliefs about FA and assessment in mathematics in general. Four hundred twenty eight lower secondary school students, that is, grade 7, grade 8, and grade 9 students participate in the study. A questionnaire focused on the beliefs about FA and assessment in mathematics in general was administrated to all participants.

3.2 Questionnaire

As we referred, the present study focuses on the first part of the whole survey. This part includes students' questionnaire about their beliefs towards FA and the concept of assessment generally. This questionnaire focuses on six axes.

The first axis investigates students' beliefs about the purpose of assessment. The statement "Assessment defines my good skills in mathematics" is an example of the statements which are related to this axis.

The second axis investigates students' beliefs about feedback, which is one of the major FA techniques. This axis consists of statements that fall in three dimensions: (a) feedback given by the teacher to the student (e.g. "When my teacher, gives me continuously information about my progress, I understand the mathematical concepts better"); (b) feedback that students give to each other – peer feedback (e.g. "I prefer not to discuss my solutions in mathematics with my classmates, in order to avoid their negative comments"); and (c) feedback given by the student to the teacher (e.g. "It is necessary to say my questions that I have about the course to the teacher at the end of the course"). The feedback among students includes the students' beliefs about peer-assessment, which is one of the FA techniques.

The third axis related with the use of errors in mathematics classroom both by the students and teachers and among the students. Students' beliefs about the use of errors by themselves were measured using statements such as "Correcting my mistakes alone, I understand the mathematical concept better".

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Similarly, students' beliefs about the use of errors in mathematics by the teacher were investigated using statements such as "The teacher should use my mistakes in order to help me to overcome my difficulties in mathematics". Other statements such as "When I discuss my mistakes with my classmates, I have more motivation to participate in the lesson" were used for investigating the students' beliefs about the use of errors among the students. This dimension of the third axis includes students' beliefs about the peer-assessment technique in relation to the formative use of errors.

The fourth axis consists of statements that investigate students' beliefs about self-assessment, which is another FA technique. One of the statements of this axis is the following: "Self-assessment does not help me to face my difficulties in mathematics".

The fifth axis 5 includes statements that investigate students' beliefs about sharing learning goals with students and defining success criteria. This axis includes statements such as "When I am assessed in mathematics, I prefer to be aware about what my teacher expects to do" or "When I am aware about the goals of the course, I participate more in the lesson".

The last axis consists of statements that investigate students' beliefs about summative assessment and grades. This axis includes statements such as "Through the test I can see my difficulties in mathematics" or "To be succeeded in mathematics means to have good grades in the progress report".

The participants completed the questionnaire using Likert scale from 1 to 4 (e.g. 1-absolutely disagree, 2-disagree, 3-agree, 4-absolutely agree).

4. Results

The analysis of the data collected in this study were conducted using the computer software Classification Hiérarchique, Implicative et Cohésitive (C.H.I.C.) (Gras et al., 2008), which gives the implicative diagrams and the similarities diagrams. This analysis in the present study indicates a hierarchical similarity between groups of variables (Lerman, 1981). In this article we analyse the axis about the use of error (third axis), because it is connected to the MWS and a probable suitable MWS of the students and of the teachers, as the students perceive it. The similarity groups appear in an ascending manner as a function of their strength. Thus, the similarity groups are represented in a hierarchically constructed similarity diagram, which allows us to study and interpret groups of items based on resemblance of performance characteristics. This analysis aims to answer the following question: *"How students' beliefs about the mathematical errors can describe a probable appropriate (suitable) MWS?"*

Figure 2 shows the similarity relations based on students' consistency in their beliefs towards the use of error in mathematics. This similarity diagram includes twenty-five variables which are related with the errors in mathematics in general and in particular it focuses on three dimensions of the use of error: students' beliefs about the use of error in mathematics a) by the students, b) by the teacher and c) among the students.

In total, two groups of variables were identified in the similarity diagram for this axis of the questionnaire (Figure 2). Following, we analyse each groups of variables separately.

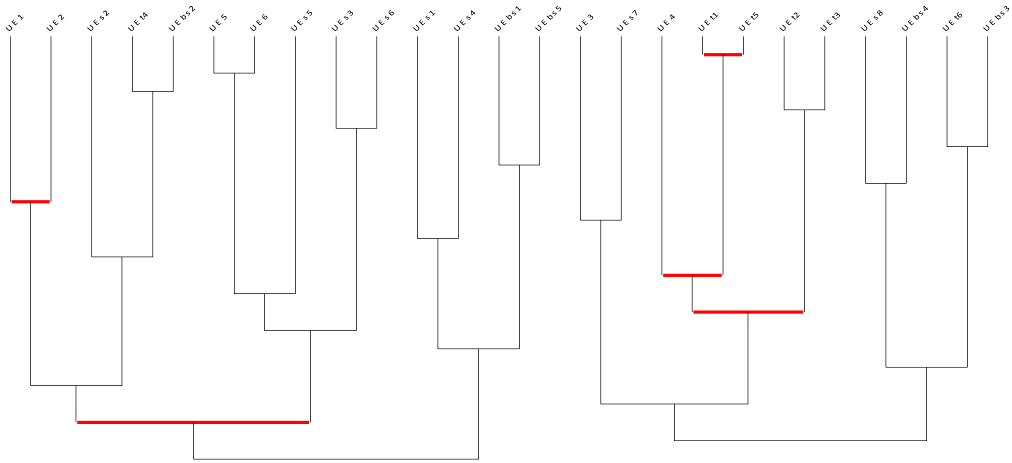

Figure 2. Similarity diagram about students' beliefs towards the use of mathematical errors.

The first group of variables consists of fourteen variables (UE1, UE2, UEs2, UEt4, UEbs2, UE5, UE6, UEs5, UEs3, UEs6, UEs1, UEs4, UEbs1, UEbs5). In this group, most of the variables are grouped due to the fact that they are related with the students' beliefs about the use of errors by themselves and between them, and not by the teacher. At a first glance, we observe the strongest similarity relationship between the variables UE5 and UE6. According to these variables, the errors in mathematics indicate a) that the students have to try harder (UE5) and b) the students' weaknesses in the particular mathematical area (UE6). Both of the statements express the utility of the errors in mathematics. The next strong similarity relationship is observed between the variables UEt4 and UEbs2. The first one supports that "the teacher has to correct students' mistakes on the whiteboard", while the second one argues that "the students feel more confident when they correct their mistakes with their classmates, because they realize that all make mistakes". This pair of variables is linked with the variable UEs2 which expresses the belief that "the teacher has to be with me when I correct my mistakes in mathematics". This similarity relationship between these three variables indicates the students' beliefs about the others' help in the correction of the errors in mathematics. Another strong relation is found between the variables UEs3, UEs6 and UEs5. All these variables refer to the use of errors by the students. In particular, the statement UEs3 describe the belief that "it is helpful the teacher to highlight students that they have errors in mathematics, but to leave them to find it by themselves". In addition, the statement UEs6 supports that "when the students correct their mistakes by themselves, it helps them to identify their weaknesses in mathematics". As a result, the variable UEs5 is connected with the both variables above, because all of them indicate that students prefer to correct their mistakes by themselves (UEs5). Less strong similarity relationship, but significant, is observed between two variables which are related with the errors in mathematics, in general. More specifically, these variables argue that if a student has errors in mathematics a) then he/she deserves a low grade (UE1) and b) means that he/she didn't study sufficiently (UE2). The variables UEbs1 and UEbs5 form another strong similarity relationship. Both variables are related with the effect that their classmates have on them, when they discuss with them their mistakes in mathematics. According to these variables, the students a) are more motivated to participate in the lesson (UEbs1) and b) have better understanding about their errors in mathematics (UEbs5) when they discuss with their classmates. As regard to the rest two variables, they are linked due to the fact that both of them refer to the use of error by themselves. According to these variables "the students have better understanding of a mathematical concept when they correct their mistakes by themselves" (UEs1), so "they prefer to correct their mistakes by themselves rather than by their classmates on the whiteboard" (UEs5).

The second cluster is formed by eleven variables (UE3, UEs7, UE4, UEt1, UEt5, UEt2, UEt3, UEs8, UEbs4, UEt6, UEbs3) which are mostly related with students' beliefs about the use of error by the teacher. The strongest similarity relationship is observed between the variables UEt1 and UEt5. This similarity relationship is significant and it was expected because both variables are related with the teachers' role in the use of error in mathematics. According to these variables, students believe that "it is important their teacher verify that they have understood their mistakes after correcting their work in mathematics" (UEt1) and this means that "the teacher has to use the errors of the students in order to help them to overcome their difficulties in mathematics" (UEt5). This pair of variables presents a less strong, but significant similarity relationship with the belief that "if I have errors in mathematics means that I didn't understand the mathematical concept" (UE4). This relation was expected due to the fact that the belief supported in statement UE4 implies the beliefs in the statements UEt1 and UEt5. All the above variables are linked with the variables UEt2 and UEt3 significantly. According to these variables "the teacher has to use the students' errors in order to design the next lesson in mathematics" (UEt2) because the students support that "they have better understanding of a mathematical concept when their teacher explains them their mistakes in an activity" (UEt3). The explanation about the similarity relationship between the five aforementioned beliefs, is that they give information about the teacher's role towards the use of students' errors in mathematics in order to help the students to overcome their difficulties and understand the particular mathematical content. The above group of the five variables presents a weak similarity relationship with the variables UE3 and UEs7. This pair of variables is connected due to the fact that both of them refer to the way of teaching. In particular, the statement UE3 argues that "the errors indicate that the teacher uses inappropriate ways of teaching", while the statement UEs7 supports that "when I correct my mistakes alone I can't have better understanding of the mathematical concept". Furthermore, another weak similarity relationship is observed between two pairs of variables. The first one consists of the variables UEt6 and UEbs3, while the second one is formed by the variables UEs8 and UEbs4. An explanation exists about the connection between the variables UEt6 and UEbs3. Both variables are related with the affective domain of students. According to the first one (UEt6) "the students don't like their teacher to comment their mistakes in the whole class". Similarly, the statement UEbs3 supports students' beliefs that "they don't like discuss their mistakes in mathematics with their classmates in order to avoid their negative comments". As regards the variables UEs8 and UEbs4, the connection between them wasn't expected, because there isn't any relation between them. However, the variable UEbs4 is related with the aforementioned pair of variables (UEt6, UEbs3) because it gives information about the effect of the errors in the affective domain of students ("I feel uncomfortable when I discuss my mistakes in groups").

As regard the table below (Table 1), the students support that "the teacher has to correct their mistakes on the whiteboard", "…to use their errors in order to help them to overcome their difficulties", "…to verify that they have understood their mistakes after correcting their work in mathematics". In addition, according to the table, students' beliefs about the use of error by the teacher are positive. The aforementioned statements are the three most positive regarding to the teachers' use of error in mathematics.

Table 1
Frequency, mean, and standard deviation of students' answers to the statements about the use of mathematical error

Statements-Variables	No answer	Absolutely disagree	Disagree	Agree	Absolutely agree	Mean	Standard deviation
Use of mathematical error		1	2	3	4		
If I have errors in mathematics, I deserve a low grade. (UE1)	19	121	176	79	33	2.37	1.68
If I have errors in mathematics means that I didn't study sufficiently. (UE2)	18	55	114	164	77	2.91	1.57
Errors indicate that the teacher uses inappropriate teaching ways. (UE3)	18	151	142	70	47	2.32	1.71
If I have errors in mathematics means that I didn't understand the mathematical concept. (UE4)	17	78	156	124	53	2.63	1.59
Errors in mathematics indicate that I have to try harder. (UE5)	38	17	55	170	148	3.67	1.84
Errors in mathematics identify my weaknesses in the particular mathematical content. (UE6)	31	26	79	186	106	3.38	1.77
Use of mathematical error by the student							
Correcting my mistakes alone, I have better understanding about the mathematical concept. (UEs1)	15	80	153	118	62	2.62	1.54
Teacher use to be with me when I correct my mistakes in mathematics. (UEs2)	23	46	155	140	64	2.89	1.69
It is helpful the teacher to highlight me that I have error in mathematics, but to leave me to find it by own/alone. (UEs3)	33	46	92	168	89	3.24	1.89
I prefer to correct my mistakes alone rather than on the whiteboard by my classmates. (UEs4)	16	105	160	84	63	2.51	1.62
I prefer to correct my mistakes alone rather than on the whiteboard by the teacher. (UEs5)	17	126	156	73	56	2.42	1.66
Correcting my mistakes alone, I determine my weaknesses in mathematics. (UEs6)	31	56	116	149	76	3.08	1.89
Correcting my mistakes alone, I can't gain	17	53	135	146	77	2.86	1.55

better understanding about the mathematical concept. (UEs7)							
Correcting my mistakes alone, I can't identify my weaknesses in mathematics. (UEs8)	21	50	125	149	83	2.96	1.65
Use of mathematical error by the teacher							
It is important my teacher verify that I have understood my mistakes after correcting my work in mathematics. (UEt1)	6	22	50	175	175	3.27	1.08
The teacher has to use our errors in order to design the next lesson in mathematics. (UEt2)	18	39	99	162	110	3.10	1.54
I have better understanding of a mathematical concept when my teacher explains me my mistakes in an activity. (UEt3)	15	34	63	176	140	3.23	1.42
The teacher has to correct our mistakes on the whiteboard. (UEt4)	22	39	53	160	154	3.36	1.60
The teacher has to use my errors in order to help me to overcome my difficulties in mathematics. (UEt5)	15	36	58	134	185	3.34	1.43
I prefer my teacher not to comment my mistakes in the whole class. (UEt6)	17	71	135	105	100	2.82	1.62
Use of mathematical error between the students							
I am more motivated to participate during the lesson when I discuss my mistakes with my classmates. (UEbs1)	16	50	125	166	71	2.86	1.50
I feel more confident when I correct my mistakes with my classmates, because I realize that all make mistakes. (UEbs2)	16	47	92	181	92	3.00	1.49
I don't like to discuss my mistakes in mathematics with my classmates in order to avoid their negative comments. (UEbs3)	11	109	151	99	58	2.43	1.45
I feel uncomfortable when I discuss my errors in groups. (UEbs4)	17	93	160	105	53	2.55	1.61
I have better understanding about my errors in mathematics when I discuss them with my classmates. (UEbs5)	13	59	146	156	54	2.69	1.42

5. Conclusions

Despite the fact that the present study doesn't present a concrete didactical situation of one mathematical concept, it is related to the importance of the role of the teacher and the interactions between the students in the creation of an appropriate Mathematical Working Space. It is also related to efficient personal Mathematical Working Spaces in the classroom in relation to the process of FA, as this occurs through the students' beliefs. In fact, in our paper we concentrate on the use of mathematical error by teachers and students. And this is because research in mathematical education has been prolific in the interpretations of students' errors.

As we described above, based on the similarity diagram, we can support that two different groups of students exist. The first one believes that a suitable use of the errors by the teacher is significant for students, in order to have better understanding about a mathematical concept. On the other hand, a large proportion of students do not believe in the proper use of the error from the teacher. This happens because the errors are frequently used by the teacher in order to give a low grade in the test or the exams, etc. This argument emerges from the first cluster of the similarity diagram. This cluster consists mostly of variables which are not related with the use of error by the teacher. Thus, the variables in the same cluster don't engage the interactions between the teacher and the students.

Furthermore, taking into account the mean of each statement, as it is presented in the table, we can conclude to three observations about the student's beliefs for a suitable/appropriate MWS. Firstly, the students argue that the most important for them is the teacher's use of their errors in mathematics through different ways in order to help them overcome their difficulties and gain better understanding about the mathematical content. Some examples of different ways of the use of error by the teacher are the following: correction on the whiteboard, the design of the next lesson, verification of the understanding of the errors, focus on the errors for helping students to overcome their difficulties. In addition, the students have positive beliefs about the use of error between them. More specifically, they strongly believe that they "feel more confident when they correct their mistakes with their classmates, because they realize that all make mistakes". They also believe that they are more motivated to participate in the lesson and they do not feel uncomfortable when they discuss their mistakes with their classmates. As regards the better understanding about their mistakes in mathematics, the students' beliefs are not clear.

The second conclusion is that the students are willing to discuss their errors in mathematics with their classmates. The third observation from the table is related with the students' beliefs about the use of error by themselves. Here, students' beliefs are less positive. According to the strongest belief of the students, "it is helpful the teacher to highlight the students that they have error in mathematics, but to leave them to find it by themselves". They also support that they do not have better understanding about the mathematical concept, when they correct their mistakes by themselves, so they prefer to correct their mistakes on the whiteboard by their classmates or their teacher rather than by themselves. The students' views about the rest of the statements are not clear.

In conclusion, a suitable MWS for the lower secondary school students includes the teacher's use of error firstly, then the interaction between the students and lastly, the use of error by each student separately, in his/her personal Mathematical Work Space. Through the examination of the students' beliefs it is clear that they give less emphasis on the formative use of errors in their personal MWS. In fact, the suitable MWS as defined by the role of the teacher has a more important role in the process of exploiting their errors. Therefore, teaching should be focussed on how to strengthen the formative use of errors by the students either in isolation or in cooperation, in order to enhance their work in their personal MWS. In this way the students will be able to perform in a more autonomous way the different kind of genesis as defined by the MWS model, with the guidance of the teacher as defined in the suitable MWS.

6. Discussion about the MWS

In general, the formative assessment is connected with the semiotic and discursive genesis of the MWS. More specifically, when the teacher or the students give feedback each other, they use a variety of semiotic means in order to communicate their ideas and solve the students' errors in the context of the formative assessment. The gestures, the glances, the speech, and the different types of representations (symbolic, picture, verbal) are some semiotic means that are used during the feedback based on the formative assessment. The speech, that is, the verbal representation is used for the mathematical errors' correction and it is connected both with the discursive genesis and the reference workspace, when the subjects give feedback about the mathematical errors based on the mathematics' theory (rules, properties, and theorem).

The questionnaire that was administered in the present study is related to the genesis of MWS, because through this, we can investigate students' beliefs and conceptions about the teachers' role and the possible interactions between both teacher and students, and also among students. For example, students' beliefs about the teachers' role are investigated using statements such as "After an assessment, my teacher should give each student different tasks, in order to help him/her to identify his/her good skill in mathematics" or "The teacher should correct our mistakes in the whiteboard". Furthermore, the above statements fall into the second and third axis, which measure students' beliefs about feedback given by the teacher to students and the use of error by the teacher respectively. Students' beliefs about the interactions between the teacher and the students are also investigated in the second and third axis of the questionnaire. More specifically, there are statements that investigate students' beliefs about the feedback given by the teacher to students and reverse, while there are statements that look for the students' beliefs about the feedback that students give each other. Both the first kind of feedback and the second one are related with the interaction between the teacher and the students, while the third one is related with the interaction among the students. Similarly, there are three dimensions for the axis of the use of error: the interaction between the teacher and the students, the interaction among the students in the use of error and the students' engagement with their own errors.

In addition, the questionnaire gives information about the students' beliefs in three different levels which can describe the diversity of MWS regarding to assessment (and FA) in the school context: (a) the reference, (b) the appropriate, and (c) the personal MWS (Kuzniak, 2011). Assessment intended by the teacher/curriculum is described in the reference of the MWS, which must be fitted out in an appropriate MWS (that is, appropriate FA techniques and in particular appropriate use of mathematical error), to enable an effective implementation in a classroom where each student works within his/her personal MWS.

In general, students' beliefs about the process of the learning arise by the questionnaire. More specifically, the students express their beliefs and conceptions about the different techniques which take place in their classroom during the process of teaching and learning mathematics. This phenomenon probably falls into the epistemological plane of the MWS model, because when the emphasis is on the processes of students' learning in a didactic situation, this epistemological plan can be considered as an epistemological environment (Coutat & Richard, 2011). On the other hand some researchers insist on the cognitive nature of the beliefs (Goldin, 1999; Mc Leon, 1992).

Gómez-Chacón, Romero Albaladejo, and García López (2016) investigated students' beliefs and how these help or impede students to transit from one kind of genesis to another. Their study was based on the following two traditional categories of attitude: attitudes towards mathematics (when the object of the attitude is mathematics itself) and mathematical attitudes (where the object of the attitude concerns the mathematical processes and activities).

In our case we believe that the coordination of the two planes is necessary in order to able to interpret the relations of students' beliefs and the MWS. In fact, in this article we tried to find traces (indices) related to MWS and to formative assessment in mathematics. Classroom assessment in mathematics education is a complex interactive process between teachers and learners and it has a crucial role in teaching and therefore, it may contribute to the improvement of learning (Veldhuis & van den Heuvel-Panhuizen, 2014). The importance of the use of formative assessment in a mathematics classroom lies in the continuous feedback that can be provided between the teacher and the learners. This feedback comes from the students who give teacher information about their understanding and their misconceptions in order to help the teacher to decide how to modify his/her teaching plan and adapt it according to the students' needs. The formative assessment refers to all students, independently of their learning level. However, it would be interest for a future research, the use of formative assessment to be investigated in students with different learning level in order to propose some guidelines about the more efficient use of the formative assessment in accordance with the students' abilities.

In addition, regarding to the model of MWS, our study presents some limitations. First of all, the model of MWS does not include particular considerations in the affective domain in mathematical education. However, the present study does not refine (or complement) the model with such considerations. As a result, there is no fine articulation between some affective variables and the components of the model in this study. We believe that more studies must be done concerning this issue in order to enrich the model of MWS with an affective dimension.

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